Working Paper No. 350

Contract N00014-85-K-0143

FINITE APPROXIMATIONS TO A ZERO-SUM GAME WITH INCOMPLETE INFORMATION

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January 1986

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Abstract : In this paper, we investigate a scheme for approximating a two-person zero-sum game G of incomplete information by means of a natural system G_{mn} of its finite subgames. The main question is : For large m and n, is an optimal strategy for G_{mn} necessarily an ϵ-optimal strategy for G?

Section 1. Introduction.

To formalize our idea of approximating a two-person zero-sum game of incomplete information by its subgames, we introduce what we shall call a game structure. A game structure is a system of the form $(\Omega, U, \mathcal{F}_m, \mathcal{G}_n)_{m,n=1}^{\infty}$. Here $\Omega = (\Omega, \mathcal{B}, P)$ is a probability space, $U = (U_{ij} : i = 1,\ldots,M; j = 1,\ldots,N)$ is a matrix of random variables on Ω (the payoff matrix), and $\mathcal{F}_m$ and $\mathcal{G}_n$ are sub-σ-fields of the σ-field $\mathcal{B}$ such that $\mathcal{F}_{m+1} \supseteq \mathcal{F}_m$ and $\mathcal{G}_{n+1} \supseteq \mathcal{G}_n$. We put $\mathcal{F} = \mathcal{F}_\infty$ = the σ-field generated by $\bigcup_m \mathcal{F}_m$, $\mathcal{G} = \mathcal{G}_\infty$ = the σ-field generated by $\bigcup_n \mathcal{G}_n$.

For $m,n = 1,2,\ldots,\infty$, let G_{mn} be the two-person, zero-sum game in which a strategy for player I is an $\mathcal{F}_m$-measurable $\alpha: \Omega \to S^M$, and a strategy for player II is a $\mathcal{G}_n$-measurable $\beta: \Omega \to S^N$. (Here S^M is the simplex $\{ x \in R^M : \sum_i x_i = 1, x_i \geq 0 \}$.) If player I plays α and player II plays β, then the payoff to I is $\Gamma(\alpha,\beta) = E(\sum_{ij} U_{ij}\, \alpha_i\, \beta_j)$. Thus in the game G_{mn}, $\mathcal{F}_m$ and $\mathcal{G}_n$ embody the information available to I and II, respectively. If $\mathcal{F}_m$ and $\mathcal{G}_n$ are finite, then G_{mn} is a finite approximation to the game $G = G_{\infty\infty}$.

By standard minimax theorems [Sion], each game G_{mn} (and a fortiori G) has a saddle point. Let V_{mn} denote the value of the game G_{mn} to player I, and let $V = V_{\infty\infty}$.

If $\mathcal{F}_m$ and $\mathcal{G}_n$ are finite, then the game G_{mn} is, at least in principle, solvable by finite methods. The question we shall study is : To what extent is an optimal strategy for G_{mn} a useful substitute for an optimal strategy for G? An ideal result along these lines would be

(1) Fix $\epsilon > 0$. Suppose that, for $m,n = 1,2,\ldots$, α^{mn} is an optimal strategy for I in G_{mn}. Then, for all sufficiently large m and n, α^{mn} is an ϵ-optimal strategy for I in G.

As we shall see, (1) is, alas, in general false. The best we can do is a weaker version of (1) (Theorem 1), and a special case of (1) (Theorem 2). We shall state these

theorems presently. For a strategy α for player I in the game G, let $\mathrm{Val}_n(\alpha) = \inf_{\beta} \Gamma(\alpha,\beta)$, where β ranges over $\mathcal{G}_n$-measurable strategies for II. (Thus if α is $\mathcal{F}_m$-measurable, then $\mathrm{Val}_n(\alpha)$ is the value to I of the strategy α in the game G_{mn}.) We shall write $\mathrm{Val}_G(\alpha)$ for $\mathrm{Val}_{\infty}(\alpha)$.

Theorem 1. For m,n = 1,2,..., suppose that α^{mn} is an optimal strategy for player I in G_{mn}, and that $\mathcal{F}_m$ and $\mathcal{G}_n$ are finite σ-fields. Then $\lim_{m\to\infty} \lim_{n\to\infty} \mathrm{Val}_G(\alpha^{mn}) = V$. Moreover, this convergence is uniform in the choices α^{mn} of optimal strategies , i.e., $\lim_{m\to\infty} \lim_{n\to\infty} \inf_{\alpha\in A(m,n)} \mathrm{Val}_G(\alpha) = V$, where $A(m,n)$ is the set of strategies optimal for player I in G_{mn}.

Theorem 2 says that, under an additional hypothesis, (1) does hold. This hypothesis, which we shall call (M), is a version of the "continuity of information" assumption first used in [Milgrom-Weber]. (M) says roughly that the joint probability on $\mathcal{F}$ and $\mathcal{G}$ is absolutely continuous with respect to the product probability on $\mathcal{F} \times \mathcal{G}$. A precise statement of (M) will be found in Sec. 2.

Theorem 2. Assume (M) holds. If, for m,n = 1,2,..., α^{mn} is an optimal strategy for I in G_{mn}, then $\lim_{\substack{m\to\infty \\ n\to\infty}} \mathrm{Val}_G(\alpha^{mn}) = V$, uniformly in the choices α^{mn} of optimal strategies.

In Sec. 2, we shall prove all of these various claims.

Section 2. Results.

We first present an example which shows that assertion (1) of the introduction does not hold in general.

Example : A game structure in which (1) fails.

Let Ω be the interval [0,1) with Lebesgue measure, M=N=2, and the payoff $U_{ij} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$ for i,j = 1,2 (independent of ω). For m = 1,2,..., let $\mathcal{F}_m = \mathcal{G}_m =$ the σ-field on [0,1) generated by the partition

$\{ [(k-1)/2^m , k/2^m) : k = 1,2,...,2^m \}$. Thus $\bar{F} = \bar{G} =$ the Borel σ-field on Ω. It is easy to see that, for all m and n, $V_{mn} = 0$, and in G_{mn} the players have the optimal strategies $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1/2$, for all $\omega \in \Omega$.

For finite $m > 1$, consider the game $G_{m,m-1}$. The strategy $\underline{\alpha}^{m,m-1}$ given by

$$\underline{\alpha}_1^{m,m-1} = \begin{cases} 1 & \text{if } \omega \in [(n-1)/2, n/2] , \text{ n odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{\alpha}_2^{m,m-1} = 1 - \underline{\alpha}_1^{m,m-1}$$

is easily seen to be optimal for player I in the game $G_{m,m-1}$, i.e., $Val_{m-1}(\underline{\alpha}^{m,m-1}) = 0$. On the other hand, $Val_G(\underline{\alpha}^{m,m-1}) = -1$; $\underline{\alpha}^{m,m-1}$ is a very poor strategy for player I in G. Thus in any system $(\alpha^{mn} : m,n = 1,2,...)$ of optimal strategies for player I in which $\alpha^{m,m-1} = \underline{\alpha}^{m,m-1}$ for all $m > 1$, $\lim_{\substack{m \to \infty \\ n \to \infty}} Val_G(\alpha^{mn}) < V$.

We shall next prove Theorems 1 and 2. We first require a series of lemmas. Our first lemma is a special case of Theorem 2 of [Blackwell- Dubins].

Lemma 2.1 Let (X_k) be a uniformly bounded sequence of random variables, and suppose that $X_k \to X_\infty$ a.s. as $k \to \infty$. Then $E(X_k \mid \bar{G}_k) \to E(X_\infty \mid \bar{G})$ a.s. as $k \to \infty$.

Our second lemma computes $Val_n(\alpha)$.

Lemma 2.2 Fix a strategy α for I in G, and fix $n \in \{1,2,...,\infty\}$. Define the random variable ξ by : ξ = the $j \in \{1,2,...,N\}$ which minimizes $E(\sum_i U_{ij}\, \alpha_i \mid \bar{G}_n)$. In case of a tie, for definiteness, take the least such j. Then , for all strategies β for II in $G_{\infty n}$,

(i) $E(\sum_i U_{i\xi}\, \alpha_i) \le \Gamma(\alpha,\beta)$, so

(ii) $Val_n(\alpha) = E(\min_j E(\sum_i U_{ij}\, \alpha_i \mid \bar{G}_n))$.

Proof : (i) immediately implies (ii), so we prove (i). Let β be a strategy for II in $G_{\infty n}$, that is, a $\bar{G}_n$ - measurable $\beta : \Omega \to S^N$. Then, since $\sum_j \beta_j = 1$, we have

$$E(\sum_i U_{i\xi}\, \alpha_i \mid \tilde{G}_n) \quad \le \quad \sum_j \beta_j\, E(\sum_i U_{ij}\, \alpha_i \mid \tilde{G}_n) \quad = \quad E(\sum_{ij} U_{ij}\, \alpha_i\, \beta_j \mid \tilde{G}_n) \text{ a.s.}$$

since β is $\tilde{G}_n$-measurable. Taking expected values on both sides yields (i).

Lemma 2.3 Let (α^k) be a sequence of strategies for I in G, and suppose that $\alpha^k \to \alpha$ a.s. as $k \to \infty$. *Then*

(i) for fixed $n = 1,2,\ldots,\infty$, $\mathrm{Val}_n(\alpha^k) \to \mathrm{Val}_n(\alpha)$ as $k \to \infty$, and

(ii) $\mathrm{Val}_k(\alpha^k) \to \mathrm{Val}_G(\alpha)$ as $k \to \infty$.

Proof: First note that applying (ii) in a system where $\tilde{G}_n = \tilde{G}_{n+1} = \ldots = \tilde{G}_\infty$ yields (i), so (i) is a special case of (ii). To prove (ii), let $X_k = \sum_i U_{ij}\, \alpha_i^k$ in lemma 2.1; then we have $E(\sum_i U_{ij}\, \alpha_i^k \mid \tilde{G}_k) \to E(\sum_i U_{ij}\, \alpha_i \mid \tilde{G})$ a.s. as $k \to \infty$, so by dominated convergence $E(\min_j E(\sum_i U_{ij}\, \alpha_i^k \mid \tilde{G}_k)) \to E(\min_j E(\sum_i U_{ij}\, \alpha_i \mid \tilde{G}))$. By lemma 2.2(ii), we are done.

Lemma 2.4 (i) For $m,n = 1,2,\ldots,\infty$, $\lim_{m\to\infty} V_{mn} = V_{\infty n}$ and $\lim_{n\to\infty} V_{mn} = V_{m\infty}$.
(ii) $\lim_{\substack{m\to\infty\\ n\to\infty}} V_{mn} = V$.

Proof : Fix n, and let α be an optimal strategy for I in $G_{\infty n}$. Now for $m = 1,2,\ldots$, put $\alpha^m = E(\alpha \mid \tilde{F}_m)$. Thus α is a legal, though likely not optimal, strategy for I in G_{mn}. We have

$$\mathrm{Val}_n(\alpha^m) \le V_{mn} \le V_{\infty n} .$$

By lemma 2.1, $\alpha^m \to \alpha$ a.s. By lemma 2.3(i), $\lim_{m\to\infty} \mathrm{Val}_n(\alpha^m) = \mathrm{Val}_n(\alpha) = V_{\infty n}$. By the inequality directly above, we infer $\lim_{m\to\infty} V_{mn} = V_{\infty n}$. By symmetry, we also have $\lim_{n\to\infty} V_{mn} = V_{m\infty}$ for all m. This proves (i). Finally, it is easy to see that $V_{m\infty} \le V_{mn} \le V_{\infty n}$. Claim (ii) now follows by letting $m, n \to \infty$.

We are now ready to prove Theorem 1.

Proof of Theorem 1 : *Suppose that* α^{mn} *is an optimal strategy for player* I in G, for $m,n = 1,2,\ldots$. We claim that, for $m = 1,2,\ldots$, $\lim_{n\to\infty} \mathrm{Val}_G(\alpha^{mn}) = V_{m\infty}$. To prove this, fix m. We shall show that every subsequence of the sequence $\mathrm{Val}_G(\alpha^{m\cdot})$ has in turn a subsequence which converges to $V_{m\infty}$. Indeed, since each α^{mn} is $\tilde{F}_m$ - measurable, $\tilde{F}_m$ being a finite σ-field, by the Bolzano-Weierstrass theorem every subsequence of $\alpha^{m\cdot}$ has

a (pointwise) convergent subsequence; thus we may assume that $\alpha^{mn} \to \alpha^m$ as $n \to \infty$. By lemma 2.3(ii), then, $Val_n(\alpha^{mn}) \to Val_G(\alpha^m)$ as $n \to \infty$. By hypothesis, $Val_n(\alpha^{mn}) = V_{mn}$, so in fact $V_{mn} \to Val_G(\alpha^m)$ as $n \to \infty$. Thus by lemma 2.4(i), $Val_G(\alpha^m) = V_{m\infty}$. On the other hand, since $\alpha^{mn} \to \alpha^m$ as $n \to \infty$, by 2.3(i) we also have $Val_G(\alpha^{mn}) \to Val_G(\alpha^m) = V_{m\infty}$. This proves our claim.

Now by another use of lemma 2.3(i), $\lim_{m\to\infty} \lim_{n\to\infty} Val_G(\alpha^{mn}) = V$. To prove the "moreover" clause in Theorem 1, let (ε_{mn}) be a sequence of numbers which converges to 0 as $m,n \to \infty$. For all m,n, note that there exists $\alpha^{mn} \in A(m,n)$ such that $Val_G(\alpha^{mn}) - \varepsilon_{mn} \leq \inf_{\alpha \in A(m,n)} Val_G(\alpha) \leq Val_G(\alpha^{mn})$. The "moreover" clause follows at once.

We now consider Theorem 2. We must first discuss hypothesis (M).

$\bar{F} \times \bar{G}$ is the σ-field on $\Omega \times \Omega$ generated by sets of the form $S \times T$, where $S \in \bar{F}$ and $T \in \bar{G}$. Let Q and R be the probability measures on $(\Omega \times \Omega, \bar{F} \times \bar{G})$ defiined by

$$Q(A) = P(\{\omega : (\omega,\omega) \in A\}) \qquad \text{and}$$
$$R(A) = \int\int_{\{(\omega,\eta) \in A\}} 1\, P(d\omega)\, P(d\eta) \qquad \text{for } A \in \bar{F} \times \bar{G}.$$

We now state assumption (M).

(M) Q is absolutely continuous with respect to R, that is, for all $A \in \bar{F} \times \bar{G}$, if $R(A)=0$, then $Q(A)=0$.

Assumption (M) is a version of a hypothesis introduced in [Milgrom-Weber]. It is easy to see that (M) is satisfied either if $\bar{F}$ and $\bar{G}$ are independent (in which case Q=R), or if either $\bar{F}$ or $\bar{G}$ is atomic.

Lemma 2.5 Suppose (M) is satisfied. Then if (X_k) is a uniformly bounded sequence of $\bar{F}$-measurable random variables which converges weakly to X_∞, and if Z is any bounded random variable, then

i) $E(X_k Z \mid \bar{G}) \to E(X_\infty Z \mid \bar{G})$ a.s. and

ii) $E(X_k Z \mid \bar{G}_k) \to E(X_\infty Z \mid \bar{G})$ a.s. as $k \to \infty$.

Proof : First note that, since $E(E(X_k Z \mid \bar{G}) \mid \bar{G}_k) = E(X_k Z \mid \bar{G}_k)$, by lemma 2.1, (i) implies (ii) . Next, note that we may assume without loss of generality that Z is measurable in $\bar{F} \vee \bar{G}$ (the -field generated by $\bar{F} \cup \bar{G}$) . This is because $E(X_k Z \mid \bar{G}) =$

$E(X_k \cdot E(Z \mid \tilde{F} \vee \tilde{G}) \mid \tilde{G})$, so we may replace Z by $E(Z \mid \tilde{F} \vee \tilde{G})$ if necessary. We shall therefore prove (i), assuming that Z is $\tilde{F} \vee \tilde{G}$ - measurable.

Since Z is $\tilde{F} \vee \tilde{G}$ - measurable, there exist a bounded $\tilde{F}$-measurable random variable $\hat{X}$, a bounded $\tilde{G}$-measurable random variable $\hat{Y}$, and a bounded, Borel measurable function $f : R \times R \to R$ such that $Z = f(\hat{X}, \hat{Y})$.

By (M) and the Radon-Nikodym theorem, there exists a bounded function $g : \Omega \times \Omega \to R$ such that, for all $A \in \tilde{F} \times \tilde{G}$, $P(\{\omega : (\omega,\omega) \in A\}) = \int_A g(\omega,\eta) P(d\omega) P(d\eta)$. It follows by standard methods that, for any vector **X** of $\tilde{F}$-measurable random variables, any vector **Y** of $\tilde{G}$-measurable random variables, and any Borel-measurable $h : R^2 \to R$, we have

$$(*) \qquad E(h(X,Y) \mid \tilde{G})(\eta) = \int_\Omega h(X(\omega), Y(\eta))\; g(\omega,\eta)\; P(d\omega) \qquad \text{a.s. } [\eta].$$

Now by (*) we have

$$E(X_k Z \mid \tilde{G})(\eta) = \int_\Omega X_k(\omega)\; f(\hat{X}(\omega), \hat{Y}(\eta))\; g(\omega,\eta)\; P(d\omega) \;\; \text{a.s.}[\eta].$$

Since, by assumption, $X_k \to X_\infty$ weakly, we have $E(X_k Z \mid \tilde{G})(\eta) \to E(X_\infty Z \mid \tilde{G})(\eta)$ a.s., as desired.

In exact analogy to lemma 2.3, we have

Lemma 2.6 Assume that (M) holds. If (α^k) is a sequence of strategies for I in G which converges weakly to a strategy α, then

(i) for fixed $n = 1,2,\ldots,\infty$, $Val_n(\alpha^k) \to Val_n(\alpha)$, and

(ii) $Val_k(\alpha^k) \to Val_G(\alpha)$ as $k \to \infty$.

Proof of Theorem 2: Suppose that α^{mn} is an optimal strategy for player I in G_{mn}, for all finite m and n. We shall prove that every sequence (m_k,n_k) of pairs of integers such that $m_k \to \infty$ and $n_k \to \infty$ has a subsequence (m'_k,n'_k) such that $Val_G(\alpha^{m'_k,n'_k}) \to V$ as $k \to \infty$. To conserve notation, let us write α^k for $\alpha^{m'_k,n'_k}$ and V_k for $V_{m'_k n'_k}$. By weak compactness, we may choose the sequence (α^k) to converge weakly to a strategy α as $k \to \infty$. By lemma 2.6(ii), $Val_{n'_k}(\alpha^k) \to Val_G(\alpha)$ as $k \to \infty$. By assumption, $Val_{n'_k}(\alpha^k) = V_k$, and by lemma 2.4(ii), $V_k \to V$; thus $Val_G(\alpha) = V$.

On the other hand, since (α^k) converges weakly to α, by lemma 2.6(i), $Val_G(\alpha^k) \to Val_G(\alpha) = V$ as $k \to \infty$. Uniformity follows just as in the proof of Theorem 1. This completes the proof of Theorem 2.

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